APPLICATION OF LINEAR PROGRAMING TO EXTREMAL PROBLEMS OF THE CONTROL THEORY

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The solution of problems on the optimal control of linear systems reducible to the L-problem of moments is described. This involves reducing the problem in question to one in linear programing.

The linear programing method makes it possible to reduce automatic extremum search time. The general computational scheme is illustrated by solving several model problems.

1. Let a controlled system be described by the vector differential Eq.

$$dz/dt = Az + bu\ . \tag{1.1}$$

Here z denotes the n-dimensional vector of the phase coordinates of the controlled object and u a scalar function describing the controlling force.

Let us consider the following problems.

Problem A. To find the function $u^\circ(t)$ (the optimal control) which satisfies restriction $|u^\circ(t)| \leqslant 1$ and brings system (1.1) from the given initial position z_0 to the origin in the smallest possible time T^0.

Problem B. To find the optimal control $u^\circ(t)$ which in a given time T brings system (1.1) from the state z_0 to the state $z(T)$ in such a way that

$$J(u) = \max\left\{\max_\tau |u(\tau)|, \theta\int_0^T |u(\tau)|\,d\tau\right\} = \min \qquad (\theta = \text{const}) \tag{1.2}$$

Problem C. To find the optimal control $u^\circ(\tau)$ which brings system (1.1) from the initial position z_0 to the origin in the shortest time T^0 in such a way that the restriction

$$\int_0^T |u(\tau)|\,d\tau = 1 \tag{1.3}$$

imposed on the control function is fulfilled.

The solutions of the above problems, all considered from the common standpoint of the L-problem of moments, are obtained in [1 and 2] and are as follows:

The optimal control for Problem A is

$$u^\circ(\tau) = \operatorname{sign}\left(\sum_{i=1}^n l_i^\circ h_i(\tau)\right) \tag{1.4}$$

where the numbers l_i° ($i = 1,\ldots, n$) are the solution of the problem

$$\min_l \int_0^T \left|\sum_{i=1}^n l_i h_i(\tau)\right| d\tau = 1, \qquad \sum_{i=1}^n l_i c_i = 1 \tag{1.5}$$

The optimal control for Problem B is

$$u^{\circ}(\tau)=\frac{1}{\alpha}\operatorname{sign}\sum_{i=1}^{n} l_i^{\circ}h_i(\tau) \quad \text{for} \quad \tau\in\Delta^{\circ}$$
$$u^{\circ}(\tau)=0 \qquad\qquad \text{for} \quad \tau\overline{\in}\Delta^{\circ} \tag{1.6}$$

where the numbers l_i° and the systems Δ° of the segments $[\tau_k, \tau_{k+1}]$ on $[0, T]$ are determined by the solution of the problem

$$\min_l \max_{\Delta}\int_{\Delta}\left|\sum_{i=1}^{n} l_i h_i(\tau)\right| d\tau=\alpha, \qquad \sum_{i=1}^{n} l_i c_i=1$$
$$\operatorname{mes}_{\Delta}=\min[\theta^{-1}, T] \tag{1.7}$$

The optimal control for Problem C is

$$u^{\circ}(\tau)=\sum_{j=1}^{r^{\bullet}}\mu_j\delta(\tau-\tau_j), \qquad \sum_{j=1}^{r}|\mu_j|=1 \tag{1.8}$$

where the symbol $\delta(\tau)$ is a pulse delta function and τ_j are the instants at which the function $[l_1^{\circ}h_1(\tau)+\ldots+l_n^{\circ}h_n(\tau)]$ reaches its maximum value on the segment $[0, T^{\circ}]$; the numbers l_1° are the solution of the problem

$$\min_l\left(\max_{\tau}\left|\sum_{i=1}^{n} l_i h_i(\tau)\right| \quad \text{for} \quad 0\leqslant\tau\leqslant T^{\circ}\right)=1, \qquad \sum_{i=1}^{n} l_i c_i=1 \tag{1.9}$$

In Formulas (1.4) to (1.9) we have

$$h_i(\tau)=\sum_{j=1}^{n} f_{ij}(-\tau)\,b_j, \qquad c_i=-Z_{i0}$$

where $f_{ij}(t)$ are the elements of the fundamental matrix $F(t)$ of homogeneous system (1.1).

In actual computation of optimal controls it is necessary to solve problems (1.5), (1.7), and (1.2). This can be done by numerical methods. The usual way of finding $\min_l$ in problems (1.5) and (1.7) is by the method of steepest descents [3 and 4]. The latter method is applied in Problem A to the function

$$\rho_A(l_1,\ldots,l_{n-1})=\int_0^T\left|g_n(\tau)+\sum_{i=1}^{n-1} l_i g_i(\tau)\right| d\tau \tag{1.10}$$

$$g_i(\tau)=h_i(\tau)+\frac{c_i}{c_n}h_n(\tau), \qquad g_n(\tau)=\frac{1}{c_n}h_n(\tau), \qquad (i=1,\ldots,n-1) \qquad (c_n\neq 0)$$

for a fixed T.

In Problem B the method of steepest descents is applied to the function

$$\rho_B(l_1,\ldots,l_{n-1}=\int_{\Delta(l)}\left|g_n(\tau)+\sum_{i=1}^{n-1} l_i g_i(\tau)\right| d\tau \tag{1.11}$$

under the assumption that the system $\Delta(l)$ of segments $[\tau_k, \tau_{k+1}]$ which yields the max in (1.7) has already been chosen.

In automatic search for the extrema of functions (1.10) and (1.11) by the method of steepest descents or by some other local search (e.g. the gradient or the relaxation) method one is faced with difficulties occasioned by the structural properties of these functions. There are very frequent cases where the functions ρ_A and ρ_B are so structured that changes in some of the variables produce relatively small changes in the values of the functions (this occurs, for example, with surfaces of the "trench" type with steep sides and a very mildly sloping floor). Search for the extrema of such functions involves rapid breakup of the operating interval; this slows down the search considerably or else stalls the computer in some secondary "mild depression".

The use of the nonlocal search method (also known as "trench" method) described by Gel'fand [5] expedites the search process. However, the total search time is still large.

The time required to find the extrema of functions (1.10) and (1.11) can be decreased substantially by reducing the problems of minimizing the functions ρ_A and ρ_B to certain problems of linear programming.

We note that the proposed computational scheme is in a sense similar to the convex programing methods developed by Pshenichnyi [6 and 7].

2. Let us break down the segment $[0, T]$ into m equal parts at the points $\tau_j = j\Delta\tau$ ($j = 0,\dots, m$). For a sufficiently large m we can write (1.10) in the form

$$\rho_A \approx \rho_A{}^* = \Delta\tau \sum_{j=1}^{m} \left| \sum_{i=1}^{n-1} l_i g_i(\tau_j) + g_n(\tau_i) \right| \tag{2.1}$$

Let us consider the system of linear functions

$$y_j(l) \equiv \sum_{i=1}^{n-1} l_i g_i(\tau_j) + g_n(\tau_j) \qquad (j = 1, \dots, m) \tag{2.2}$$

The problem of minimizing $\rho_A{}^*$ is then equivalent to the following problem of minimizing a convex piecewise-linear function:

$$v(l) = \sum_{j=1}^{m} | y_j(l_1, \dots, l_{n-1}) | \tag{2.3}$$

The problem of minimizing (2.3) can be reduced to a linear programing problem [7 and 8].

To effect this reduction we introduce the additional variables $x_1,\dots, x_m$, setting

$$| y_j(l) | \leqslant x_j \qquad \text{or} \qquad x_j + y_j(l) \geqslant 0, \qquad x_j - y_j(l) \geqslant 0 \tag{2.4}$$

The problem of minimizing (2.3) is now equivalent to the following linear programing problem of minimizing the function

$$L = x_1 + \dots + x_m \tag{2.5}$$

under restrictions (2.4).

In fact, let $L'' = \min L$ under restrictions (2.4) and let it be attained at the point (l'', x''); let $v' = \min v$ and let it be attained at the point l'. The Eqs. $x_j'' = |y_j(l'')|$ are clearly fulfilled at the point (l'', x''), since $x_j'' \geqslant |y_j(l'')|$ by virtue of (2.4), and since the corresponding x_j can be reduced in searching for min L in the absence of the equality sign for certain j. Hence,

$$L'' = \sum_{j=1}^{m} x_j'' = \sum_{j=1}^{m} | y_j(l'') | \geqslant v' \tag{2.6}$$

Now let $x_j' = |y_j(l')|$. The point (l', x') then satisfies restrictions (2.4), so that

$$L'' \leqslant \sum_{j=1}^{m} x_j' = \sum_{j=1}^{m} | y_j(l') | = v' \tag{2.7}$$

From (2.6) and (2.7) we conclude that $L'' = v'$ and l'' from the solution (l'', x'') of problem (2.4), (2.5) is also the solution of problem (2.3).

As a typical problem of linear programing, minimization of function (2.5) can be effected by the simplex method.

We note that the linear programing problem for finding the minimum of the function ρ_B can be constructed as described above.

Now let us consider the problem of finding the extremum of the function

$$\rho_c(l_1, \ldots, l_{n-1}) = \min_l\left(\max_\tau \left| g_n(\tau) + \sum_{i=1}^{n-1} l_i g_i(\tau)\right|\right) \quad (0 \leq \tau \leq T) \tag{2.8}$$

This is the problem to which we can reduce finding the minimum of the function in the left-hand side of Eq. (1.9) for a fixed T. Determination of this minimum is one of the steps in the solution of problem C.

We assume once again that the segment $[0, T]$ has been broken up into m equal parts at the points $\tau_j = j\Delta\tau$ ($j = 0, \ldots, m$).

Now let us consider the vector function $\phi(l, \tau)$ with the components

$$\varphi_j(l, \tau_j) = \left| g_n(\tau_j) + \sum_{i=1}^{n-1} l_i g_i(\tau_j)\right| \qquad (j = 1, \ldots, m) \tag{2.9}$$

and the set M of vectors S,

$$s_j = \{\underbrace{0, \ldots, 0}_{j}, 1, \ldots, 0\} \qquad (j = 1, \ldots, m)$$

For a sufficiently large m problem (2.8) can be approximated by the problem

$$\rho_c(l) \approx \rho_c^* = \min_l\,(\max_s\,(\varphi\,(l, \tau),\ s)])\ \ (s \in M) \tag{2.10}$$

Here the symbol (ϕ, s) denotes the scalar product of the vectors ϕ and s.

Let us consider the system of linear functions

$$y_j(l) = g_n(\tau_j) + \sum_{i=1}^{n-1} l_i g_i(\tau_j) \qquad (j = 1, \ldots, m) \tag{2.11}$$

The problem of minimizing the piecewise-linear convex function

$$v(l) = \max_s\,(\varphi\,(l, \tau),\ s) \qquad (s \in M) \tag{2.12}$$

is the Chebyshev problem of approximating system (2.11). This can also be reduced to a linear programing problem [8].

To this end we introduce the new variable x_0, setting

$$\varphi_j\ (l, \tau_j) \leq x_0 \qquad (j = 1, \ldots, m) \tag{2.13}$$

The equivalent linear programing problem can be formulated as follows.

We are to minimize the function

$$L = x_0 \tag{2.14}$$

under the restrictions

$$x_0 + g_n(\tau_j) + \sum_{i=1}^{n-1} l_i g_i(\tau_j) \geq 0, \qquad x_0 - g_n(\tau_j) - \sum_{i=1}^{n-1} l_i g_i(\tau_j) \geq 0 \tag{2.15}$$

Let us show that the solution of problem (2.14), (2.15) is at the same time the solution of problem (2.10). Let $L' = \min x_0$ under restrictions (2.15) and let it be attained at the point (l', x).

It is then clear that

$$L' = x_0' = \max_s\,(\varphi\,(l', \tau),\ s) \geq \min_l\ [\max_s\,(\varphi\,(l, \tau),\ s] = \rho_c^* \qquad (s \in M)$$

On the other hand, if l'' is some Chebyshev point of system (2.11), we have $\phi_j(l'', \tau_j) \leq \rho_c^*$ ($j = 1, \ldots, m$). This means that the point $(l'', x_0 = \rho_c^*)$ satisfies restrictions (2.15). But since L' is the minimum value of x_0 under restrictions (2.15), it cannot exceed ρ_c^*, i.e. $L' \leq \rho_c^*$.

From these two inequalities we conclude that $\rho^* = L'$. Hence,

$$\rho_c^* = \max_s\,(\varphi\,(l', \tau),\ s) = \min_l\ [\max_s\,(\varphi\,(l, \tau),\ s)] \qquad (s \in M)$$

i.e. the point l' is the solution of problem (2.10) for $\rho_c^* = x_0'$.

In conclusion we note that problem (2.14), (2.15) can be solved by the simplex method.

3. Example 1. Let us solve the problem of shortening the time required to bring a gyrocompass to a given meridian [9]. The gyroscope motion is described by Eqs.

$$z_1^{\cdot} = q_{12}z_2 + q_{13}z_3 + u(\tau), \quad z_2^{\cdot} = q_{21}z_1, \quad z_3^{\cdot} = q_{32}z_2 + q_{33}z_3 \tag{3.1}$$

Here

$$q_{12} = 3.74 \cdot 10^{-2}, \qquad q_{13} = 2.32 \cdot 10^{-2}, \qquad q_{21} = -4.11 \cdot 10^{-5},$$
$$q_{32} = -1.5 \cdot 10^{-3}, \quad q_{33} = -1.5 \cdot 10^{-3}$$

The problem of bringing system (3.1) to the origin in a fixed time T under the condition of minimality of the norm $\|u\|$ of the controlling function ($\|u\| = \max_\tau |u(\tau)|$, $0 \leqslant \tau \leqslant T$) reduces to the problem [1] of finding

$$\min_l \int_0^T \left| \sum_{i=1}^{3} l_i h_i(\tau) \right| d\tau = \rho(\tau) = \frac{1}{\|u\|} \quad (l_1c_1 + l_2c_2 + l_3c_3 = 1) \tag{3.2}$$

Here

$$T = 1800 \text{ сек}, \quad c_1 = 2.87 \cdot 10^{-2}, \quad c_2 = 1{,}39 \cdot 10^{-2}, \quad c_3 = -1.01 \cdot 10^{-2}$$

$$h_1(\tau) = a_{11}e^{x(1-\tau)} + e^{\varepsilon(T-\tau)}(b_{11}\cos\omega(T-\tau) - c_{11}\sin\omega(T-\tau))$$
$$h_2(\tau) = a_{21}e^{x(T-\tau)} + e^{\varepsilon(T-\tau)}(b_{31}\cos\omega(T-\tau) - c_{31}\sin\omega(T-\tau))$$
$$h_3(\tau) = a_{31}e^{x(T-\tau)} + e^{\varepsilon(T-\tau)}(b_{21}\cos\omega(T-\tau) - c_{31}\sin\omega(T-\tau))$$
$$x = -0.8824 \cdot 10^{-3}, \quad \varepsilon = -0.3088 \cdot 10^{-3}, \quad \omega = 0.9481 \cdot 10^{-3}$$
$$a_{11} = -4.438 \cdot 10^{-1}, \quad a_{21} = -0.0207, \quad a_{31} = 0.0503$$
$$b_{11} = 1.444, \quad b_{21} = 0.0207, \quad b_{31} = -0.0503$$
$$c_{11} = -0.0572, \quad c_{21} = 0.0559, \quad c_{31} = -0.0304$$

On eliminating l_3 we can rewrite (3.2) as

$$= \min_{l_1, l_2} \int_0^T |l_1g_1(\tau) + l_2g_2(\tau) + g_3(\tau)|\, d\tau \tag{3.3}$$

$$g_1(\tau) = h_1(\tau) - \frac{c_1}{c_3}h_3(\tau) \tag{3.4}$$

$$g_2(\tau) = h_2(\tau) - \frac{c_2}{c_3}h_3(\tau), \quad g_3(\tau) = \frac{1}{c_3}h_3(\tau)$$

The functions g_1, g_2, and g_3 are plotted in Fig. 1. The linear programing problem for solution (3.3) can be written as

$$L = x_1 + \ldots + x_m \tag{3.5}$$

under the restrictions

$$x_j + l_1g_1(\tau_j) + l_2g_2(\tau_j) + g_3(\tau_j) \geqslant 0$$
$$x_j - l_1g_1(\tau_j) - l_2g_2(\tau_j) - g_3(\tau_j) \geqslant 0 \tag{3.6}$$
$$(j = 1, \ldots, m)$$

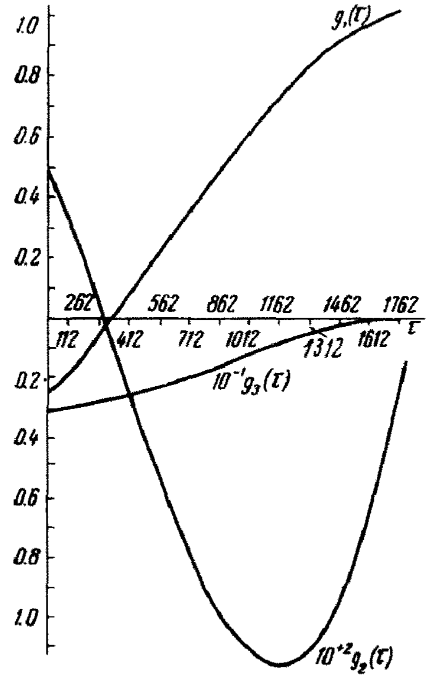

Fig. 1

In order to make the solution of the above optimal problem more accurate we must choose our m sufficiently large. This imparts a large dimensionality to linear programing problem (3.5), (3.6). For example, for $m = 50$ the initial matrix for solving the problem by the simplex method has the dimensionality (98 × 196). The dimensionality of the linear programing problem can be reduced without diminishing the accuracy of the initial problem by computing integral (3.3) as a sum of trapezoid areas. In this case the segment $T = 1800$ sec breaks down into 14 unequal parts. The average values of the ordinates

$$g_1(\tau_j^*),\ g_2(\tau_j^*),\ g_3(\tau_j^*)$$

of functions (3.4) are given in the table.

j	$\Delta\tau_j$	$g_1(\tau_j^*)$	$g_2(\tau_j^*)\cdot 10^{-2}$	$g_3(\tau_j^*)$
1	187.5	—0.1868	0.385	—3.001
2	300	0.0089	—0.102	—2.651
3	300	0.2723	—0.6447	—2.077
4	150	0.4745	—0.975	—1.5765
5	150	0.603	—1.11	—1.223
6	75	0.695	—1.162	—0.962
7	75	0.751	—1.163	—0.795
8	75	0.804	—1.138	—0.636
9	75	0.852	—1.1084	—0.489
10	75	0.8935	—0.997	—0.354
11	75	0.9295	—0.876	—0.237
12	75	0.959	—0.718	—0.140
13	75	0.981	—0.52	—0.0667
14	112.5	0.9948	—0.205	—0.0181

Problem (3.5), (3.6) for the values of $g_1(\tau_j)$, $g_2(\tau_j)$, and $g_3(\tau_j)$ given in the table was solved by the simplex method. This yielded

$$l_1^0 = -1.143, \qquad l_2^0 = -162.5$$

The resulting value of ρ^0 was $\rho^0 = 2228$.

Hence, the required optimal control in accordance with (1.4) turned out to be

$$u^0(\tau) = 0.448\cdot 10^{-3}\,\mathrm{sign}\,(-1.143\,g_1(\tau) - 162.5\,g_2(\tau) + g_3(\tau)) \quad (3.7)$$

The motion of system (3.1) under the action of control (3.7) is represented by the plots of the functions $Z_1(\tau)$, $z_2(\tau)$, and $z_3(\tau)$ in Fig. 2.

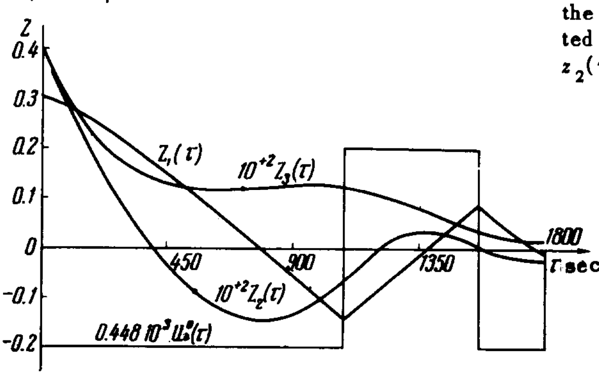

Fig. 2

E x a m p l e 2. Let us consider the problem of coming to rest of a linear oscillator,

$$z_1' = z_2, \quad z_2' = -z_1 + u(\tau);$$
$$z_1(0) = -1, \quad z_2(0) = 0 \quad (3.8)$$

in the time $T = 1.18$ under the condition of a minimum momentum

$$J(u) = \int_0^T |u(\tau)|\,d\tau \quad (3.9)$$

In accordance with (1.9) we can solve this problem by finding

$$\rho^\circ = \min_{l_2}(\max_\tau |h_1(\tau) + l_2 h_2(\tau)|, \quad 0 \leqslant \tau \leqslant 1.18) \quad (3.10)$$

where

$$h_1(\tau) = -\sin\tau, \quad h_2(\tau) = \cos\tau$$

We can solve Problem (3.10) by reducing it to a linear programing problem. To this end we break down the segment $[0, T]$ into nine segments by means of the points $\tau_j = j\,0.131$ ($j = 0,\ldots, 9$). The equivalent linear programing problem can be formulated as follows. We are to find the minimum of the form

$$L = x_0 \quad (3.11)$$

under the restrictions

$$x_0 + h_1(\tau_j) + l_2 h_2(\tau_j) \geqslant 0, \quad x_0 - h_1(\tau_j) - l_2 h_2(\tau_j) \geqslant 0 \quad (j = 0, \ldots, 9) \quad (3.12)$$

Problem (3.11), (3.12) was solved by the simplex method. The optimal value of the parameter l_2 is $l_2^\circ = 0.668$. The function

$$g(\tau) = |-\sin\tau + 0.668\cos\tau| \quad (3.13)$$

is plotted in Fig. 3. From this plot we see that the function $g(\tau)$ attains its maximum values in the segment $[0, 1.18]$, i.e. the values $\rho^0 = 0.668$ at the points $\tau_1 = 0$, $\tau_2 = 1.18$. Hence,

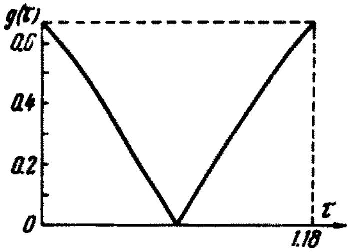

Fig. 3

Fig. 4

in accordance with (1.8) the optimal control is of the form

$$u^0(\tau) = \mu_1\delta(\tau - 0) + \mu_2\delta(\tau - 1.18) \qquad (3.14)$$

To determine the quantities μ_1 and μ_2 appearing in Formula (3.14) we make use of the stipulation that the representing point must reach the origin at the instant $\tau = 1.18$. This condition implies the following Eqs.:

$$1 = -\int_0^{1.18} \sin\tau\,[\mu_1\delta(\tau - 0) + \mu_2\delta(\tau - 1.18)]\,d\tau$$

$$0 = \int_0^{1.18} \cos\tau\,[\mu_1\delta(\tau - 0) + \mu_2\delta(\tau - 1.18)]\,d\tau$$

Carrying out the integrations, we obtain

$$\mu_1 = 0.415, \ \mu_2 = -1.083 \qquad (3.15)$$

The optimal control with allowance for (3.15) is shown in Fig. 4.

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